THE ASTRONOMICAL JOURNAL VOLUME 80, NUMBER 9 SEPTEMBER 1975

Eclipses by an elliptical torus

Robert E. Wilson*

Department of Astronomy, University of South Florida, Tampa, Florida
Goddard Institute for Space Studies, New York, New York
Department of Physics and Astronomy, University of Florida, Gainesville, Florida

(Received 28 April 1975; revised 17 June 1975)

The geometry of a torus having elliptical meridian sections is discussed in regard to its eclipsing properties when viewed at arbitrary inclinations. Eclipses involving the hole horizon as well as the outer horizon are considered. Various special cases of such a torus include those of a thin ring or disk, an ellipsoid of revolution, and a section of a right circular cylinder. Thus the relations given here may be used in place of a number of special schemes used previously for these particular cases, as well as for the case of a general torus of finite thickness. A simple method is given by means of which one can decide if an arbitrary point in space is or is not eclipsed by the torus. Leading up to this procedure, a general horizon condition is derived and the basic equations of the problem are listed, as are quadrant rules for the surface coordinates of the torus. Certain basic equations might be used to derive analytic eclipse functions for special cases, such as eclipses of limb-darkened spheres, although this has not been done in the present paper. Major simplifications are made possible by the definition of an auxiliary ellipsoid, points on whose surface are mapped in one-to-one correspondence into surface points of the torus. Finally, some discussion of practical computational problems is given, and a FORTRAN subroutine, TORUS, is briefly described.

INTRODUCTION

IN recent years, explanations for the peculiarities of a number of unusual binary star systems have been given in terms of disks or rings of circumstellar matter around one component of the binary [e.g., for ϵ Aur, Huang (1965, 1974a, 1974b); Cameron (1971); Wilson (1971); for β Lyr, Huang (1963); Wilson (1974); Kriz (1974); for BM Orionis, Hall (1971); Wilson (1972)]. The geometrical thicknesses of these disks are in some cases negligible and in others fairly great. Sometimes there is a central opening (the disk is a ring) and sometimes not. In one case the disk model is a simple ellipsoid of revolution, in another a "red blood cell," and in another a section of a right circular cylinder. To some it may seem unnecessary to attain much mathematical rigor in the treatment of such disks, since the models are intended only as approximations to the true form of the circumstellar matter distribution. However, it must be remembered that if an impersonal technique, in the form of the method of differential corrections, is to be applied to the observations, as surely seems desirable, then it is necessary to form *derivatives* of the apparent system brightness with respect to the various model parameters (usually by numerical means). Naturally, if the mathematical properties of the disk, with regard to eclipses, have been defined only qualitatively, it will be impossible to compute such derivatives and therefore impossible to evaluate the model parameters impersonally.

Observations of another unusual binary, specific results on which will be published later, seem to require a disk having both of the two most troublesome features found in the above cases. That is, the observations require a "disk" with a central hole (or at least a central depression) *and* a finite thickness. The form is thus that of a torus or "donut." Therefore, it seemed worthwhile to develop a general procedure for computing the effects of eclipses by a torus, including both the outer horizon and the inner or hole horizon. In most cases, the hole horizon can eclipse only the component located within the torus. Such a procedure will actually be more valuable than one might first suspect, because special cases of a general torus with elliptical meridian cross section cover all of the types of disk models mentioned earlier. That is, we can make the thickness negligible and find a thin ring, set the hole radius to zero and find a thin disk, or we can make the radius to the center of the elliptical meridian section equal to zero and find an ellipsoid of revolution. We can even make the equatorial axis of the elliptic meridian section equal to zero and find a section of a right circular cylinder. Thus the procedure which follows can be used for any of the simple figures of revolution one might reasonably encounter in binary-star models.

I. THE GEOMETRY

Define a coordinate system with origin O at the center of the torus, polar angle θ (colatitude), and longitude angle ϕ, with $\phi=0$ corresponding to the line of centers between binary components. Point O will coincide with the center of one component of the binary system. Let this line of centers also be the x axis of a right-handed, rectangular coordinate system whose y axis lies in the orbit plane and whose z axis is normal to the orbit plane. The equatorial plane of the torus will coincide with the x,y plane (binary-orbit plane). A plane for $\phi=$constant will cut the torus in an ellipse

---

* NAS–NRC senior Research Associate, 1972–1974. Present address: Department of Astronomy, University of South Florida, Tampa, Florida.

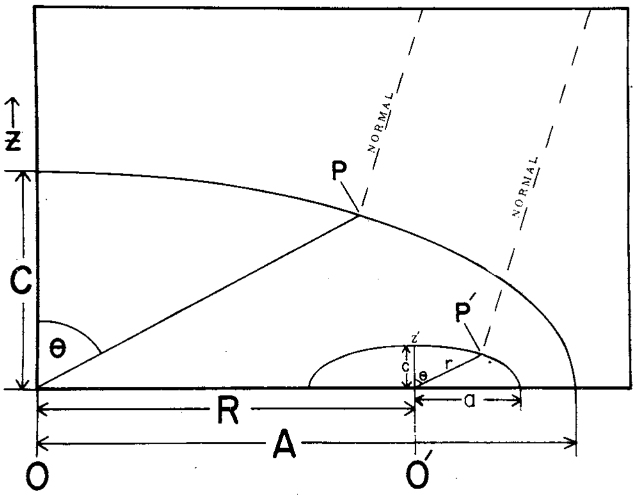

FIG. 1. Torus geometry at a fixed value of ϕ. Meridian sections of the auxiliary ellipsoid (large quarter-ellipse) and of the torus (small half-ellipse) are shown. The centers of the auxiliary ellipsoid and of the entire torus (point O) are coincident with the center of one component of the binary system.

whose semiaxes are a, c and whose center is at O', as in Fig. 1. (Of course, there will actually be two such elliptical sections, but for a particular value of ϕ we consider only the one in the "positive" direction. The other ellipse will correspond to longitude $\phi \pm \pi$.) Further denote the distance OO' by R and the distance from O' to an arbitrary point P' on the elliptical section by r. In the same plane we can imagine a larger elliptical section (axes A, C) of the same shape and orientation and centered on O, points (P) on whose circumference are in one-to-one correspondence with points (P') on the first ellipse. That is, (cf. Fig. 1) points P and P' are related through the condition

$$\angle Z'O'P' = \angle ZOP = \theta.$$

Note that the normals to P and P' in the plane ϕ = constant coincide with the corresponding normals to the ellipsoid and torus, since both are figures of revolution. Since the plane normals at P and P' are parallel, the same can be said of the three-dimensional surface normals at these points, and thus they form the same angle with the line of sight for an infinitely distant observer. For this observer, a point P' will be on the horizon of the torus if and only if the corresponding point P is on the horizon of our large ellipsoid. Further, since this condition holds at any value of ϕ, we should be able to develop a general horizon test in terms of θ, ϕ, which will be the same for the torus as for our ellipsoid of revolution having axes A, C and centered on O. We shall take as our basic horizon definition the intersection in a right angle of a surface normal and the line of sight. Therefore, because of the hole in the torus, we include sections of horizon which would be out of view in the case of an opaque torus. It is best, therefore, to think of the horizons we shall find as those of a *transparent* torus. When dealing with an opaque torus, one may then simply discard those sections of horizon which are out of sight.

The x, y, z coordinates of an arbitrary point P' can be written as the sum or the corresponding coordinates of O' with respect to O and of P' with respect to O'. Thus,

$$\begin{aligned} x &= r \sin\theta \cos\phi + R \cos\phi, \\ y &= r \sin\theta \sin\phi + R \sin\phi, \\ z &= r \cos\theta. \end{aligned} \tag{1}$$

The projection of these into the plane of the sky (coordinates $y_{\rm sky}$, $z_{\rm sky}$; origin projected onto O) is

$$\begin{aligned} y_{\rm sky} &= -x \sin\Theta + y \cos\Theta, \\ z_{\rm sky} &= -x \cos i \cos\Theta - y \cos i \sin\Theta + z \sin i, \end{aligned} \tag{2}$$

where Θ, i are the orbital phase and inclination, respectively. Because the torus is a figure of revolution, obviously there can be no difference in the aspect which it presents at various phases. Therefore, we can, without loss of generality, fix Θ at any convenient value. We choose $\Theta = 0$, so that the insertion of Eqs. 1 and 2 yields

$$\begin{aligned} y_{\rm sky} &= r \sin\theta \sin\phi + R \sin\phi, \\ z_{\rm sky} &= -(r \sin\theta \cos\phi + R \cos\phi) \cos i + r \cos\theta \sin i. \end{aligned} \tag{3}$$

Equations 3 give the plane of sky coordinates for an arbitrary point on the torus, relative to the center of the torus. Now r can be expressed in terms of the semiaxes a, c as

$$r = ac/[c^2 + (a^2 - c^2) \cos^2\theta]^{\frac{1}{2}}, \tag{4}$$

so that Eqs. 3 become

$$\begin{aligned} y_{\rm sky} &= \frac{ac \sin\theta \sin\phi}{[c^2 + (a^2 - c^2) \cos^2\theta]^{\frac{1}{2}}} + R \sin\phi, \\ z_{\rm sky} &= \frac{ac(\cos\theta \sin i - \sin\theta \cos\phi \cos i)}{[c^2 + (a^2 - c^2) \cos^2\theta]^{\frac{1}{2}}} - R \cos\phi \cos i. \end{aligned} \tag{5}$$

As a reasonable scheme for computing the effects of eclipses by the torus, we presume to have available the $y_{\rm sky}$, $z_{\rm sky}$ (viz., Eqs. 2) coordinates of an arbitrary point on one of the binary component stars, which we will test against a horizon function (which must be consistent with Eqs. 5), to see if that point lies within the projected boundaries of the torus (outer horizon or hole horizon). However, Eqs. 5 are expressed in terms of the θ, ϕ surface coordinates on the torus, which are not related in any simple way with *arbitrary* $y_{\rm sky}$, $z_{\rm sky}$ coordinates (i.e., with points which may or may not project onto the torus).

We may, however, proceed as follows:

(1) Form the quotient of Eqs. 5, and substitute for θ in terms of ϕ from a general horizon condition, which

we shall derive. This will give $z_{\rm sky}/y_{\rm sky}$ as a function of ϕ for horizon points.

(2) Since we know the ratio $z_{\rm sky}/y_{\rm sky}$ from the location of our arbitrary, possibly eclipsed, point, we may solve the abovementioned equation for ϕ by any convenient means such as the Newton–Raphson method.

(3) Substitution of ϕ back into the horizon condition will yield θ, and (ϕ,θ) substituted into Eqs. 5 will yield the $y_{\rm sky}$, $z_{\rm sky}$ coordinates of that point on the torus horizon which lies along the direction toward the (possibly eclipsed) point. One may then readily compare the distances from the origin to the test point and to the horizon point to determine if the test point is eclipsed or not.

(4) Other variations on this basic idea are possible, although these will not be followed in detail in this paper. For example, a representative collection of horizon points might be obtained in the abovementioned way, and a smooth approximation polynomial fitted to these for later use in fast and simple eclipse calculations. Alternatively, the horizon condition in θ, ϕ might be substituted directly into Eqs. 5, yielding a parametric representation of the horizon [i.e., two equations for $y_{\rm sky}(\phi)$ and $z_{\rm sky}(\phi)$].

We now find the condition for an arbitrary point on the torus to be a horizon point. To do this, we must evaluate the scalar product of two vectors, one of which is normal to surface at P; while the other is a unit vector along the line of sight. When the scalar product is zero, P' must be on a horizon of the torus. In the x, y, z system, it can be shown that the normal vector has components

$$n_x=-\sin\theta\cos\phi,\qquad n_y=-\sin\theta\sin\phi,\qquad n_z=-\frac{a^2}{c^2}\cos\theta,\tag{6}$$

while the line-of-sight vector has components

$$l_x=\cos\Theta\sin i,\qquad l_y=\sin\Theta\sin i,\qquad l_z=\cos i.\tag{7}$$

Again we choose $\Theta=0$, so that the scalar product of Eqs. 6 and 7, equated to zero, gives

$$\sin i\sin\theta\cos\phi+\frac{a^2}{c^2}\cos\theta\cos i=0,\tag{8}$$

which may be rearranged to give

$$\tan\theta\cos\phi=-\frac{a^2}{c^2\tan i}.\tag{9}$$

Equation 9 is the condition that an arbitrary point on the surface of the torus be a horizon point.

We now form the quotient of the two Eqs. 5, substituting at the same time for θ in terms of ϕ from the horizon condition Eq. 9. We find

$$\frac{z_{\rm sky}}{y_{\rm sky}}=\left[c\left(\sin i+\frac{a^2}{c^2}\frac{\cos i}{\tan i}\right)-R\cos\phi\cos i\left(\frac{a^2}{c^2\cos^2\phi\tan^2 i}+1\right)^{\frac12}\right]\Bigg/\left[\frac{a^2}{c}\frac{\tan\phi}{\tan i}+R\sin\phi\left(\frac{a^2}{c^2\cos^2\phi\tan^2 i}+1\right)^{\frac12}\right].\tag{10}$$

If we compare the square-root factor in Eq. 10 with Eqs. 4 and 9, we can easily show that it consists of a factor c/r, which is positive by definition, and an implicit factor of $1/\cos\theta$. Therefore, the sign of the root is positive or negative according to whether $\cos\theta$ is positive or negative. One can establish the quadrant rules given in Table I by visualizing an inclined torus and noting in which θ quadrant and which ϕ quadrant the horizon will lie. Quadrant rules are necessary not only to establish the sign of $\cos\theta$, and thus of the root in Eq. 10, but also in supplying starting values for the Newton–Raphson solutions for ϕ. For the outer horizon (subscript out) the ϕ, θ quadrant rules are the same for all inclinations, but for the inner (hole) horizon we have a change of rules corresponding to the critical inclination at which one can just begin to see through the hole.

In order to carry out the Newton–Raphson solution of Eq. 10 for ϕ, we should have available the derivative $d(z_{\rm sky}/y_{\rm sky})/d\phi$. We list this derivative, which is rather complicated, in a series of subparts which are well suited to computer coding. First, we write Eq. 10 in the compressed form

$$\frac{z_{\rm sky}}{y_{\rm sky}}=\frac{\mathrm{T1}+\mathrm{T2}}{\mathrm{T3}+\mathrm{T4}},\tag{11}$$

where the meanings of T1, etc., are obvious by comparison of Eqs. 10 and 11. The derivative of Eq. 11 is,

TABLE I. Quadrant rules for ϕ, θ.

$z_{\rm sky}/y_{\rm sky}$	$\phi_{\rm out}$	$\phi_{\rm in}$ i near 90°	$\phi_{\rm in}$ i not near 90°	$\theta_{\rm out}$	$\theta_{\rm in}$ near 90°	$\theta_{\rm in}$ not near 90°
I	II	I	II	I	IV	III
II	III	IV	III	I	IV	III
III	IV	III	IV	II	III	IV
IV	I	II	I	II	III	IV

of course,

$$\frac{d\dfrac{z_{\rm sky}}{y_{\rm sky}}}{d\phi}=\frac{(\mathrm{T3}+\mathrm{T4})\dfrac{d\mathrm{T2}}{d\phi}-(\mathrm{T1}+\mathrm{T2})\left(\dfrac{d\mathrm{T3}}{d\phi}+\dfrac{d\mathrm{T4}}{d\phi}\right)}{(\mathrm{T3}+\mathrm{T4})^2}. \quad (12)$$

The derivatives of T2, T3, and T4 are given by

$$\frac{d\mathrm{T2}}{d\phi}=R\cos i\sin\phi\left[\left(\frac{a^2}{c^2\cos^2\phi\tan^2 i}+1\right)^{\frac{1}{2}}-\frac{a^2}{c^2\tan^2 i\cos^2\phi}\left(\frac{a^2}{c^2\cos^2\phi\tan^2 i}+1\right)^{-\frac{1}{2}}\right], \quad (13)$$

$$\frac{d\mathrm{T3}}{d\phi}=-\frac{a^2}{c\tan i\cos^2\phi}, \quad (14)$$

and

$$\frac{d\mathrm{T4}}{d\phi}=R\left[\frac{a^2\tan^2\phi}{c^2\cos^2\phi\tan^2 i}\left(\frac{a^2}{c^2\cos^2\phi\tan^2 i}+1\right)^{-\frac{1}{2}}+\cos\phi\left(\frac{a^2}{c^2\cos^2\phi\tan^2 i}+1\right)^{\frac{1}{2}}\right]. \quad (15)$$

II. PRACTICAL COMPUTATIONS

A FORTRAN subroutine (named TORUS) has been developed which utilizes the foregoing equations to compute a point on the outer horizon or hole horizon of a torus for an arbitrary position angle in the plane of the sky. One enters the $y_{\rm sky}$, $z_{\rm sky}$ coordinates of a point in the plane of the sky (obtained by application of Eqs. 2) and an integer which tells whether the outer horizon or hole horizon is to be found. Usually this point would be on the surface of one of the binary-component stars, although it need not be. Of course the parameters which describe the torus (R, a, c, i) must also be entered. For output we have the $y_{\rm sky}$, $z_{\rm sky}$ coordinates of that horizon point which has the same position angle as the arbitrary point. The external (calling) program can then use this information to apply an eclipse test in any one of several obvious ways. Copies of this subroutine will be provided to interested potential users.

The Newton–Raphson iterations in TORUS always converge fairly quickly, with one minor qualification. If the inclination is very close to 90° (say $i>89\overset{\circ}{.}5$) the iterations *for some sections* of the horizon (the curved "ends") may not converge because the derivative defined by Eq. 12 approaches zero as i approaches 90°, and this derivative appears as a divisor in the iterative scheme. However, one can treat these "ends" as a special case, since for $i\simeq 90°$ their shape differs very little from that of a simple ellipse. The elliptical approximation gives

$$y_{\rm sky}=\frac{c^2R\pm c[c^2R^2-(c^2+a^2K^2)(R^2-a^2)]^{\frac{1}{2}}}{c^2+a^2K^2} \quad (16)$$

and

$$z_{\rm sky}=Ky_{\rm sky}, \quad (17)$$

where K is the ratio of the arbitrary (input) coordinates [i.e., $K=(z_{\rm sky}/y_{\rm sky})_{\rm in}$].

In Eq. 16, the positive sign applies for the outer horizon, the negative sign for the hole horizon. No other qualifications on the performance of the general procedure seem needed, except that computations for the hole horizon sometimes require fairly good initial estimates for ϕ, and that a scheme must be included to enable the proper ϕ quadrant to be reentered in the event that an early iteration pushes the operating point into a neighboring quadrant.

ACKNOWLEDGMENTS

Part of this work was done during the latter part of two-years' support by the U. S. National Academy of Sciences–National Research Council, which has been very much appreciated. Figure 1 was made by Mr. L. W. Twigg.

REFERENCES

Cameron, A. G. W. (1971). Nature **229**, 178.
Hall, D. S. (1971). Proc. IAU Colloq. **15**.
Huang, S. (1963). Astrophys. J. **138**, 342.
Huang, S. (1965). Astrophys. J. **141**, 976.
Huang, S. (1974a). Astrophys. J. **187**, 87.
Huang, S. (1974b). Astrophys. J. **189**, 485.
Kriz, S. (1974). Bull. Astron. Inst. Czech. **25**, 1.
Wilson, R. E. (1971). Astrophys. J. **170**, 529.
Wilson, R. E. (1972). Astrophys. Space Sci. **19**, 165.
Wilson, R. E. (1974). Astrophys. J. **189**, 319.